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Note

Slowly Pulsating Axisymmetric Systems in General Relativity

1. Introduction

Levy has investigated gravitational induction for slowly moving but permanently axisymmetric systems using a perturbation method [1]. (See also [2, 3].)

He assumes a metric g_{ij}, which is independent of the variable x^3, i.e.,

$$g_{ij} = g_{ij}(x^0, x^1, x^2).$$

Then he chooses the coordinate curves x^0, x^1, x^2 to lie in the surfaces $x^3 =$ constant, obtaining

$$g_{\alpha 3} = 0 \qquad \text{for} \quad \alpha = 0, 1, 2.$$

Here x^1, x^2, and x^3 are to be interpreted as a generalization of cylindrical polar coordinates, and x^0 as the time coordinate,

$$x^0 = t, \qquad x^1 = r, \qquad x^2 = z, \qquad \text{and} \qquad x^3 = \phi.$$

Then the vacuum field equations $R_{ij} = 0$ reduce to seven, as the coordinate conditions lead to

$$R_{03} = R_{13} = R_{23} \equiv 0.$$

One then defines a matrix

$$a_{ij} = \begin{pmatrix} e^{2u} & 0 & 0 & 0 \\ 0 & -e^{2k-2u} & 0 & 0 \\ 0 & 0 & -e^{2k-2u} & 0 \\ 0 & 0 & 0 & -r^2e^{-2u} \end{pmatrix},$$

where both u and k depend on x^0, x^1, and x^2. Thus a_{ij} has the form of the static metric when Weyl's canonical coordinates are used. One also assumes that the matrix a_{ij} satisfies the static vacuum field equations,

$$\nabla^2 u = 0,$$

$$\frac{1}{r}\frac{\partial k}{\partial r} = \left(\frac{\partial u}{\partial r}\right)^2 - \left(\frac{\partial u}{\partial z}\right)^2,$$

$$\frac{1}{r}\frac{\partial k}{\partial z} = 2\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}.$$

Therefore a_{ij} is the Weyl metric if the motion ceases. In general, though, Levy assumes

$$g_{ij}(t, r, z) = a_{ij}(t, r, z) + h_{ij}(t, r, z),$$

and also $h_{\alpha 3} = 0$, $\alpha = 0, 1, 2$. Here h_{ij} is to be seen as a "correction" term.

Assuming that the motion is slow, then if v is a characteristic velocity of the source $| v/c | \sim O(\epsilon)$, where ϵ is a small parameter, and because $g_{ij} \to a_{ij}$ as $\epsilon \to 0$, $h_{ij} \sim O(\epsilon)$. Also,

$$\partial/\partial t \sim O(\epsilon).$$

The zero order vacuum field equations are, of course, the Weyl equations. Levy calculates the first order equations $O(\epsilon)$ and obtains two sets of field equations.

$$\text{Group I:} \quad R_{00} = R_{11} = R_{22} = R_{33} = R_{12} = 0.$$

These equations contain the five functions h_{00}, h_{11}, h_{22}, h_{33}, h_{12} as well as u, k and their space derivatives.

$$\text{Group II:} \quad R_{01} = R_{02} = 0.$$

These two equations contain only h_{01}, h_{02} as well as u, k and their time derivatives as well as space derivatives. (The tensor components thus split up into two groups, and we will for convenience denote the indices of Group I by A and those of Group II by B; thus R_A represents R_{00}, R_{11}, R_{22}, R_{33}, or R_{12} and R_B represents R_{01} or R_{02}.)

One may therefore assume that $h_A = 0$, since Group I does not contain information on the development of the system; this means $R_A \equiv 0$. Also, if one assumes that, for an isolated source, space is asymptotically flat, then the coordinate system may be identified asymptotically with flat space cylindrical polars. So in terms of the spherical polar coordinate R, where $R^2 = r^2 + z^2$, the boundary conditions are $u \sim O(R^{-1})$, $h_B \sim O(R^{-1})$, and $k \sim O(R^{-2})$. The last condition comes from the zero order field equations. Thus one can assume that in this case, as $\nabla^2 u = 0$,

$$u = \sum_{n=0}^{\infty} [A_n/R^{n+1}]\, P_n(\cos\theta),$$

where $A_n = A_n(t)$. Also k is of the form

$$k = A_0^2 f(\theta)/R^2 + \cdots.$$

Levy then examines the first order equations $R_B = 0$. He obtains an expression which by comparison with Newtonian theory can be identified as representing a Poynting vector for gravitational energy transfer. His argument for this largely rests on assuming that $\dot{A}_0 = 0$ and $\ddot{A}_1 = 0$ for an isolated system. The condition $\dot{A}_0 = 0$ represents the conservation of mass and is verified by substituting the above expressions for u and k in the first order equations.

The condition $\ddot{A}_1 = 0$ represents the conservation of momentum and can only be verified from the second order equations, which are far too complicated to obtain by hand.

2. The Programming System LAM

It was felt desirable to check Levy's hand calculations and, in addition, to obtain the second order equations, $O(\epsilon^2)$. To this end the author implemented an algebraic programming system LAM on the Stockholm IBM 360/75. LAM was developed by R. A. d'Inverno on the London ATLAS computer specifically for performing calculations in general relativity [4]. LAM is written in LISP [5], and users thus require a slight knowledge of that langugage.

The system consists of four basic packages and one application package for tensor calculations. There is a simplification package SIMP which simplifies expressions. This, applied to $A + A$, will return $2A$. There is also a differentiation package, DIFF, which performs partial differentiation. A package which can produce very readable output is called PRT. Thus $(\partial A/\partial x^2)^4$ is written as $A_2{}^4$. There is also a package for substitutions. Finally there is a package for tensor calculations called GEOM which calculates the usual quantities of Riemannian geometry starting from either the covariant metric or both the covariant and contravariant metric. For details of LAM, see [6].

Although LAM is a relatively simple algebraic system, it has managed to perform calculations in general relativity beyond the scope of other systems.

3. The Second Order Equations

Using LAM we have repeated the work of Section 1, but we have carried it to the second order, i.e., retaining only terms up to $O(\epsilon^2)$. We explicitly give the order of the h_{ij}; that is, instead of h_{ij} we write ϵh_{ij}, and, similarly, the second order correction to the metric which we call p_{ij} is given as input in the form $\epsilon^2 p_{ij}$. This is because it is much easier to program the condition $O(\epsilon^3) = 0$.

We assume a metric of the form

$$
\begin{aligned}
g_{00} &= e^{2u} + p_{00}\epsilon^2,\\
g_{01} &= h_{01}\epsilon + p_{01}\epsilon^2,\\
g_{02} &= h_{02}\epsilon + p_{02}\epsilon^2,\\
g_{03} &= 0,\\
g_{11} &= -e^{-2u+2k} + p_{11}\epsilon^2,\\
g_{12} &= p_{12}\epsilon^2,\\
g_{13} &= 0,\\
g_{22} &= -e^{-2u+2k} + p_{22}\epsilon^2,\\
g_{23} &= 0,\\
g_{33} &= -r^2\epsilon^{-2u} + p_{33}\epsilon^2.
\end{aligned}
$$

Also we have, as previously stated, that $O(\epsilon^3) = 0$ and $\partial/\partial t = O(\epsilon)$.

Since we assume that the contravariant metric may be expanded in powers of ϵ, then

$$g^{ij} = a^{ij} + h^{ij}\epsilon + p^{ij}\epsilon^2.$$

Using $(a^{ij} + h^{ij}\epsilon + p^{ij}\epsilon^2)(a_{jk} + h_{jk}\epsilon + p_{jk}\epsilon^2) = \delta_k{}^i$, one obtains $a^{ij}a_{jk} = \delta_k{}^i$. which gives $a^{ii} = a_{ii}^{-1}$, $h^{il}\epsilon = -a^{ij}h_{jk}a^{kl}\epsilon$, and $p^{il}\epsilon^2 = a^{ij}h_{jk}h_{mp}a^{km}a^{pl}\epsilon^2 - a^{ij}p_{jk}a^{kl}\epsilon^2$, Hence the contravariant components g^{ij} of the metric may be calculated.

Next the lists for g_{ij} and g^{ij} are given to the GEOM package, and the condition $O(\epsilon^3) = 0$ is implemented by using the substitution package and arranging that both ϵ^3 and ϵ^4 be replaced by zero. The condition $\partial/\partial t = O(\epsilon)$ is obtained by modifying the basic LAM package DIFF.

Because of storage problems, the Riemann tensor was first stored on tape and then special routines calculated the Ricci tensor, which in turn was placed on tape. The equations so obtained, $R_{ij} = 0$, are the vacuum field equations up to the second order. They were also obtained by d'Inverno, using the original ALAM system [4]. A big advantage, now that they were stored on tape, was that it was possible to manipulate these equations within the computer. To do this by hand would have been virtually impossible.

As Levy had conjectured [1], one finds that $\ddot{A}_1 = 0$ in the case of an isolated source and, as mentioned previously, represents the conservation of momentum. This result is obtained by substituting the multiple expansions for u and k in the second order equations. Also, comparison of our first order equations with those obtained by Levy showed that the physically interesting equations, $R_B = 0$, were the same. No check was made on $R_A = 0$, as we assumed $h_A = 0$ in the input.

Another immediate result on examining the second order terms is that once again the equations fall into the same two groups.

$$\text{Group I:} \quad R_A = 0.$$

These equations contain p_A as well as u, k, h_B and their time derivatives as well as space derivatives.

$$\text{Group II:} \quad R_B = 0.$$

These two equations contain only p_B as well as u, k and their space derivatives. Thus, in this case, one may assume $p_B = 0$, as group II does not contain information about the development of the system. Thus $R_B \equiv 0$.

It may be asked if this behavior continues for all orders. In fact, one finds that the field equations $R_A = 0$ only contain terms which are even in the quantities g_B and $\partial/\partial x^0$, whereas the field equations $R_B = 0$ only contain terms which are odd in the quantities g_B and $\partial/\partial x^0$. This can be seen if one realizes that ultimately R_{ij} consists of tensor operations on g_{ij} and its derivatives. Each time an index 0 occurs due to a contraction, it must be matched by another index 0. Thus R_B will have index 0 occurring an odd number of times in every term, whereas R_A will have index 0 occurring an even number of times in every term.

The result of this grouping of the Ricci tensor is that one may assume that g_B may be expanded as

$$g_B(\epsilon, t, r, z) = \epsilon g_B{}^1 + \epsilon^3 g_B{}^3 + \cdots, \tag{1}$$

that is, in only odd powers of ϵ. Also g_A may be expanded as

$$g_A(\epsilon, t, r, z) = g_A{}^0 + \epsilon^2 g_A{}^2 + \epsilon^4 g_A{}^4 + \cdots, \tag{2}$$

that is, in only even powers of ϵ.

4. Checking the Results

Although the two equations $R_B = 0$ are satisfied if one assumes $p_B = 0$, there are still five equations left, $R_A = 0$, each containing between 100 to 150 terms. This illustrates a common problem in algebraic computing; namely, the output is sometimes of such a size that it is unmanageable by hand.

Nonetheless one would like to check the field equations obtained, and in general relativity this can be done by investigating the contracted Bianchi identities,

$$(R_j{}^i - \tfrac{1}{2}\partial_j{}^i R)_{;i} = 0.$$

These are the integrability conditions of the Ricci tensor, and give an identity which the metric tensor must satisfy.

Unfortunately, these identities are rather long and thus, in order to keep the size of the calculation within reasonable bounds, the zero and first order field equations were used to simplify the Bianchi identities. One must pay for this, however, inasmuch as in order to then show that they are satisfied one must use the lower order field equations. There are two simplified second order Bianchi identities (of the original four, one vanishes by axial symmetry and the other may be shown to vanish due to the fact that $\partial/\partial t \sim O(\epsilon)$), one of which is

$$B1 \equiv e^{2u-2k}\left\{-\frac{\partial R_{12}}{\partial x^1} - \frac{1}{2}\frac{\partial R_{22}}{\partial x^2} + \frac{1}{2}\frac{\partial R_{11}}{\partial x^2} - \frac{1}{x^1}R_{12}\right\} - \frac{1}{2}e^{-2u}\frac{\partial R_{00}}{\partial x^2} + \frac{1}{2}\frac{1}{(x^1)^2}e^{2u}\frac{\partial R_{33}}{\partial x^2} \equiv 0.$$

The other is a similar equation.

It was thus necessary to calculate some of the derivatives of the R_{ij}. Here the calculation began to blow up since each derivative contains between 200 and 300 terms. However, after simplifying the above expression, it was found that $B1$ contains around 400 terms. (To obtain $B1$ it was found necessary to store each term on tape and call them in one by one.)

The last stage of the calculations was to reduce this expression to zero using the lower order field equations and their first and second derivatives as mentioned above. The technique used was as follows. For an equation of the form $T1 + T2 + T3 + \cdots = 0$ where $T1$ is a simple term, every occurrence of $T1$ was replaced by $-(T2 + T3 + \cdots)$ in $B1$.

Since there are six lower order equations and, with their possible derivatives, one obtains over 50 equations, it was important to pick the correct ones, and a certain amount of inspection of $B1$ was needed before each substitution. However, $B1$ was eventually reduced to 20 terms which, on inspection, was seen to be equivalent to zero. One met with the old problem of how to substitute, a quite difficult question when one is faced with so large an expression and so large a number of substitution equations. It was decided, at any stage, always to substitute for the term which appears least often in the remaining substitution equations.

By showing that the second order field equations satisfy a Bianchi identity one can be quite confident that these equations are correct.

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